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# Finite amplitude instability of time-dependent flows

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An extension of the Stuart–Watson technique for examination of the non-linear hydrodynamic instability of time-dependent flows is proposed. An example in Bénard convection is used to illustrate the method. Extensions to more general problems are indicated.

## 1. Introduction

The techniques of analysis of the stability or instability of a time-independent basic state are fairly well understood. On one hand one can use the energy method (Joseph 1965, 1966) and obtain a sufficient condition for stability against disturbances of arbitrary amplitude. On the other hand, the governing disturbance equations can be linearized; the result is a criterion for the growth or decay of infinitesimal disturbances. This criterion gives a sufficient condition that the basic state be unstable. Corrections to the linear theory which take into account weak non-linearities make the approximation uniformly valid in time (Stuart 1960*b*) and further are able to predict subcritical instabilities. These methods are particularly enlightening in the cases of convective and centrifugal instabilities where these various methods make self-consistent predictions (Davis 1969*b*).

However, the problem of the instability of a time-dependent basic state has received little attention.

The first difficulty encountered in the formulation is the decision as to a criterion for instability. Since the basic state can grow (or decay) simultaneously with the growth of a disturbance, a 'relative' criterion for instability is sometimes appropriate. These problems have been discussed by Shen (1961) and Rosenblat (1968).

Rosenblat (1968) has discussed the infinitesimal stability of inviscid Couette flow. Other analyses (Lick 1965; Morton 1957; Currie 1967) have considered infinitesimal stability of viscous flows with the assumption that the basic state is quasi-static, i.e. that the time variation of the basic state is slow compared with the growth rate of a disturbance. The validity of this assumption has been discussed by Robinson (1967). Venezian (1969) considered modulated, linearized thermal convection using perturbation theory. Foster (1965*a*) has numerically integrated the linearized disturbance equations governing an impulsively heated layer. Yih (1968) has discussed the linearized viscous flow on an oscillating plate and also the difficiencies of the quasi-static approximation. Grosch & Salwen (1968) have used Galerkin's method to treat the linearized plane Poiseuille flow with a modulated pressure gradient. Gresho & Sani (1970) have considered a time modulation of a Bénard convection layer using Galerkin's method. Conrad & Criminale (1965*a*, *b*) have used the energy method to obtain sufficient conditions for stability in some problems associated with shear and curved flows.

We here wish to make a first approach to a non-linear stability theory for time-dependent basic states. All non-linear theories presume the full knowledge of the linear theory solution which involves the solution of second-order linear equations with time-dependent coefficients. To avoid lengthy calculation we will choose a problem which does not involve this difficulty but we shall indicate the procedure in these more difficult cases. We shall avoid the decision of choosing a stability criterion by determining the time variation of *any* small enough disturbance from its initial value to its ultimate state. Having done this, one can later choose a criterion and test for stability. The determination of the behaviour of a disturbance for all time is made using an extension of the method of Stuart (1960*a*) and Watson (1960) in which the full non-linear partial differential equations which govern the problem are reduced to a single (or set of) ordinary, non-linear differential equation(s), called the amplitude equation(s).

The object of this paper is to explain the method which is illustrated in the context of Bénard convection in a horizontal fluid layer with free boundaries subject to heating from below. The time variation is introduced by oscillating the layer in the vertical direction. This is the same problem considered by Gresho & Sani (1970). In order to avoid (for the time being) the difficulties associated with the linear problem, we take the Prandtl number $P \to \infty$. This reduces the order of time differentiation from two to one and allows us to obtain closed form solutions.

In principle, the method is applicable to a whole class of stability problems with fairly general basic state time variation. No assumption is made as to the relative time scale of the basic state compared to the rate of growth of a disturbance. Generalizations are discussed.

## 2. Formulation

The following notation will be used: $d$ is the mean distance between two infinite horizontal planes that are stress free and kept at constant temperatures, the lower at $T_h$, and the upper at $T_c$ with $T_h > T_c$. $d^{-1}(T_h - T_c)$ is denoted by $\beta$. The planes bound a fluid of reference density $\rho_0$. The acceleration of gravity (taken to act vertically downward) is $g_0(1-g(t))$ and $\alpha, \nu$ and $\kappa$ are the constant coefficients of thermal expansion, kinematic viscosity and thermal diffusivity of the fluid respectively. The dimensionless horizontal co-ordinates $x$ and $y$ and vertical co-ordinate $z$ are referred to length $d/\pi$; the velocity vector $\mathbf{v} = (u, v, w)$, the temperature $T$, the time $t$ and the pressure $p$ are made dimensionless by reference to the scales $\kappa\pi/d$, $\beta d/\pi$, $d^2/\pi^2\kappa$ and $\rho_0\kappa\nu\pi^2/d^2$, respectively. We employ the Boussinesq approximation under which the governing equations become the following:

$$\left(-P^{-1}\frac{\partial}{\partial t}+\nabla^2\right)\mathbf{v}+(1-g(t))R\theta\mathbf{k}-\nabla p = P^{-1}\mathbf{v}.\nabla\mathbf{v}, \qquad (2.1\,a)$$

$$\left(-\frac{\partial}{\partial t}+\nabla^2\right)\theta - w\overline{T}_z = \mathbf{v}.\nabla\theta - (\overline{w\theta})_z, \tag{2.1b}$$

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}\right)\overline{T} = (\overline{w\theta})_z, \tag{2.1c}$$

$$u_x + v_y + w_z = 0. \tag{2.1d}$$

Here a bar indicates the horizontal average; $\overline{T}$ is the horizontal average of $T$ and $\theta = T-\overline{T}$. Subscripts denote partial derivatives, $R = \alpha\beta g_0 d^4/\pi^4\kappa\nu$ is the Rayleigh number and $P = \nu/\kappa$ is the Prandtl number.

We restrict attention to the case of two-dimensional flow and take the Prandtl number to be infinite. This allows the solutions to be obtained in closed form which is helpful in illustrating the approach.

Equations (2.1) then become, upon elimination of the pressure, the following:

$$\nabla^4 w + (1-g(t))R\theta_{xx} = 0, \tag{2.2a}$$

$$\left(-\frac{\partial}{\partial t}+\nabla^2\right)\theta - w\overline{T}_z = u\theta_x + w\theta_z - (\overline{w\theta})_z, \tag{2.2b}$$

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}\right)\overline{T} = (\overline{w\theta})_z, \tag{2.2c}$$

$$u_x + w_z = 0. \tag{2.2d}$$

We wish to solve these equations on the domain $-\infty < x < \infty, 0 \leqslant z \leqslant \pi, t \geqslant 0$. We impose the conditions on the horizontal boundaries corresponding to planar, stress-free, perfectly conducting planes.

$$w = u_z = \theta = 0 \quad \text{on} \quad z = 0, \pi \tag{2.3a}$$

and hence from (2.2$d$) that

$$w_{zz} = 0 \quad \text{on} \quad z = 0, \pi. \tag{2.3b}$$

The mean temperature satisfies

$$\overline{T}(0,t) = \pi \quad \text{and} \quad \overline{T}(\pi,t) = 0. \tag{2.3c}$$

All dependent variables are taken to be periodic in $x$ of wave-number $\alpha$.

To complete the specification of the problem, we must impose initial conditions. It has been shown (Eckhaus 1965) in the case of a steady basic state that the non-linear development of a disturbance is composed of two asymptotic time regions. In the 'inner' region fairly general initial values of a disturbance decay exponentially to zero as long as they correspond to stable modes according to linear theory; disturbances which grow according to linear theory survive into an 'outer' region where they are followed using the method of Stuart (1960$a$) and Watson (1960). These time regions emerge from an asymptotic analysis in the small parameter $(R-R_L)^{\frac{1}{2}}$ where $R_L$ is the critical value of the Rayleigh number $R$ according to linear theory. We will restrict ourselves to a single wave-number in the $x$ direction and hence only a single disturbance, the fundamental of linear theory, will grow in the 'inner' layer. We then will present the evolution of this mode in the 'outer' region. The appropriate initial condition is the following:

$$\left.\begin{aligned} w(x,z,0) &= W_0 \cos\alpha x \sin z, \\ \theta(x,z,0) &= \Theta_0 \cos\alpha x \sin z, \\ u(x,z,0) &= U_0 \sin\alpha x \cos z, \end{aligned}\right\} \tag{2.4}$$

where $\Theta_0 = W_0(1+\alpha^2)^2/(1-g(0))\,\alpha^2 R$ and $U_0 = -W_0\alpha^{-1}$ (compatibility conditions) and where we have anticipated the results of the linear theory and imposed the initial condition appropriate to the 'outer' layer. We stress that we have in effect considered an initial condition containing a finite number of modes allowable according to linear theory but only the evolution of the growing fundamental appears to the order we are considering.

## 3. Formal expansion

The basic state whose stability is being examined satisfies $\mathbf{v} \equiv 0$, $\overline{T} = T_0(z)$, $\theta = 0$ where

$$T_0(z) = -(z-\pi).$$

This solution satisfies (2.2$c$) and (2.3$c$).

The formal expansion can be motivated by considering the linearized problem apart from the initial conditions.

$$\left(\frac{\partial^2}{\partial z^2}-\alpha^2\right)^2 \hat{w}_1 - \alpha^2 R(1-g(t))\,\hat{\theta}_1 = 0, \tag{3.1$a$}$$

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}-\alpha^2\right)\hat{\theta}_1 + \hat{w}_1 = 0, \tag{3.1$b$}$$

$$\hat{w}_1 = \hat{w}_{1zz} = \hat{\theta}_1 = 0 \quad \text{on} \quad z = 0, \pi, \tag{3.1$c$}$$

where we have used that $\overline{T}_{0z} = -1$ and have separated the $x$ dependence of all dependent variables in the form $\exp(i\alpha x)$.

A solution to system (3.1) can be found in the form

$$(\hat{w}_1, \hat{\theta}_1) = (F_1(t), G_1(t)) \exp(a_0 t) \sin z, \tag{3.2$a$}$$

where $F_1$ and $G_1$ must satisfy

$$(1+\alpha^2)^2 F_1 - \alpha^2 R(1-g(t))\, G_1 = 0$$

and

$$-dG_1/dt - (a_0+1+\alpha^2)\, G_1 + F_1 = 0.$$

The solutions then are the following:

$$G_1(t) = G_1(0)\exp\left[-\frac{\alpha^2 R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right], \tag{3.2$b$}$$

$$F_1(t) = \frac{\alpha^2 R}{(1+\alpha^2)^2}(1-g(t))\,G_1(t), \tag{3.2$c$}$$

where

$$a_0 = \frac{\alpha^2}{(1+\alpha^2)^2}(R-R_L) \quad \text{and} \quad R_L = \frac{(1+\alpha^2)^3}{\alpha^2}. \tag{3.2$d,e$}$$

One notices that the solutions are composed of products of two parts. The first is $\exp(a_0 t) = \exp[(\alpha^2/(1+\alpha^2)^2)(R-R_L)t]$ which is an exponential growth whose growth rate can be made arbitrarily small by making $R$ sufficiently close to its linear theory critical value $R_L$. The second factor is one which is bounded in time whenever $g$ is integrable on $[0, \infty)$.

The non-linear theory to be developed uses the same spirit as that used for steady flows by Stuart (1960*a*) who reasoned as follows: An exponentially growing disturbance quickly invalidates the linearization. Let us replace the exponential by an unknown function $A(t)$ which grows exponentially when non-linear terms are neglected but which remains bounded for all time when modified by the non-linearities. Formal expansions are then written in powers of $A(t)$. It is also assumed that

$$dA/dt = a_0 A - a_2 A^3 + \dots .$$

The coefficients $a_i$ are chosen so as to suppress certain secular terms, called replication terms, which would make the expansion non-uniformly valid in time. By analogy, we write as follows:

$$\begin{aligned}\theta(x,z,t) = {} & [A(t)\theta_1(z,t) + A^3(t)\theta_3(z,t) + \dots]\cos\alpha x + [A^2(t)\theta_2(z,t) + \dots]\cos 2\alpha x \\ & + [A^2(t)\theta_0(z,t) + \dots] + [A^3(t)\theta_4(z,t) + \dots]\cos 3\alpha x + \dots,\end{aligned} \tag{3.3a}$$

$$\begin{aligned}w(x,z,t) = {} & [A(t)w_1(z,t) + A^3(t)w_3(z,t) + \dots]\cos\alpha x \\ & + [A^2(t)w_2(z,t) + \dots]\cos 2\alpha x + [A^2(t)w_0(z,t) + \dots] \\ & + [A^3(t)w_4(z,t) + \dots] + \dots,\end{aligned} \tag{3.3b}$$

$$\begin{aligned}\alpha u(x,z,t) = {} & [A(t)u_1(z,t) + A^3(t)u_3(z,t) + \dots]\cos\alpha x \\ & + [A^2(t)u_2(z,t) + \dots]\cos 2\alpha x + [A^2(t)u_0(z,t) + \dots] \\ & + [A^3(t)u_4(z,t) + \dots]\cos 3\alpha x + \dots,\end{aligned} \tag{3.3c}$$

$$\overline{T}(z,t) = T_0(z,t) + A^2(t)T_2(z,t) + \dots, \tag{3.3d}$$

$$\frac{dA}{dt} = a_0 A - a_2(t)A^3 + \dots, \tag{3.3e}$$

with the boundary conditions

$$\left.\begin{aligned} & \theta_i(0,t) \\ & \quad = \theta_i(\pi,t) = w_i(0,t) = w_i(\pi,t) = u_i(0,t) = u_i(\pi,t) = 0, \\ & T_0(0,t) = \pi, \\ & \qquad T_0(\pi,t) = T_{2i}(0,t) = T_{2i}(\pi,t) = 0 \quad \text{for all } t \text{ and } \quad i = 1,2,3,4,\dots\end{aligned}\right\} \tag{3.3f}$$

and the initial conditions

$$\left.\begin{aligned} \theta_1(z,0) = \Theta_0 \sin z, \quad w_1(z,0) = W_0 \sin z, \quad u_1(z,0) = U_0 \cos z, \\ T_0(z,0) = -(z-\pi); \quad \theta_i(z,0) = w_i(z,0) = u_i(z,0) = T_{2i}(z,0) = 0 \quad (i > 1).\end{aligned}\right\} \tag{3.3g}$$

These initial conditions are chosen in order to excite the most unstable mode according to linear theory.

These expansions are precisely those of Stuart's (1960*a*) theory except that the $\theta_i$, $w_i$, $u_i$ and $a_i$ are now time dependent to allow for the bounded part of the time dependence in the linearized solution not included in $A(t)$.

A word of caution is appropriate here. The expansion (3.3) is not merely 'written' but it the logical outcome of attempting a solution by successive approximations (Segel 1965). In particular, the amplitude equation (3.3$e$) is first order in time. More generally (i.e. for $P$ finite), the amplitude equation would take the form

$$(1+\alpha^2)^{-1}(1+P)^{-1}\frac{d^2A}{dt^2}+\frac{dA}{dt} = a_0A-a_2(t)A^3+\dots. \tag{3.4}$$

This is similar to that obtained by Segel & Stuart (1962) in a case when the basic state is steady.

If expansions (3.3) are substituted into (3.1), a sequence of linear inhomogeneous problems is obtained.

## 4. Solutions

The order one equations obtained above are the following:

$$\left(-\frac{\partial}{\partial t}+\frac{\partial^2}{\partial z^2}\right)T_0 = 0,$$

$$T_0(0,t) = \pi, \quad T_0(\pi,t) = 0, \quad T_0(z,0) = -(z-\pi).$$

The basic state whose stability is to be studied is thus

$$T_0(z,t) = -(z-\pi). \tag{4.1}$$

The next set of equations are those of $O(A\cos\alpha x)$, the linear theory:

$$\left(\frac{\partial^2}{\partial z^2}-\alpha^2\right)^2 w_1-\alpha^2R(1-g(t))\,\theta_1 = 0,$$

$$\left(-\frac{\partial}{\partial t}-a_0+\frac{\partial^2}{\partial z^2}-\alpha^2\right)\theta_1+w_1 = 0.$$

The solution, satisfying the boundary conditions (2.3) and initial conditions (2.4), are those of (3.2) and are repeated as follows:

$$(\theta_1, w_1) = (G_1(t), F_1(t))\sin z \tag{4.2a}$$

and

$$G_1(t) = \Theta_0\exp\left\{-\frac{\alpha^2R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right\}, \tag{4.2b}$$

$$F_1(t) = [\alpha^2R/(1+\alpha^2)^2]\,(1-g(t))\,G_1(t), \tag{4.2c}$$

where

$$a_0 = [\alpha^2/(1+\alpha^2)^2]\,(R-R_L), \tag{4.2d}$$

$$R_L = (1+\alpha^2)^3/\alpha^2. \tag{4.2e}$$

The first correction to linear theory involves terms of $O(A^2\cos 2\alpha x)$:

$$\left(\frac{\partial^2}{\partial z^2}-4\alpha^2\right)^2 w_2-4\alpha^2R(1-g(t))\,\theta_2 = 0, \tag{4.3a}$$

$$\left(-\frac{\partial}{\partial t}-2a_0+\frac{\partial^2}{\partial z^2}-4\alpha^2\right)\theta_2+w_2 = \tfrac{1}{2}(w_1\theta_{1z}-w_{1z}\theta_1), \tag{4.3b}$$

subject to the boundary conditions

$$\theta_2 = w_2 = w_{2zz} = 0 \quad \text{on} \quad z = 0, \pi \tag{4.3c}$$

and initial conditions $$\theta_2 = w_2 = 0 \quad \text{at} \quad t = 0. \tag{4.3d}$$

Using the solutions (4.2) we find that $\frac{1}{2}(w_1 \theta_{1z} - w_{1z}\theta_1) \equiv 0$.

It is easily shown subject to the zero initial conditions (4.3$d$) that

$$w_2 = u_2 = \theta_2 \equiv 0. \tag{4.4}$$

The remaining second-order terms correct the mean temperature and are $O(A^2)$:

$$\left(-\frac{\partial}{\partial t} - 2a_0 + \frac{\partial^2}{\partial z^2}\right) T_2 = \tfrac{1}{2}(w_1\theta_1)_z, \tag{4.5a}$$

with $$T_2(0,t) = T_2(\pi,t) = 0, \tag{4.5b}$$

and $$T_2(z,0) = 0. \tag{4.5c}$$

The solution is

$$T_2(z,t) = H_2(t) \exp\left\{-\frac{2\alpha^2 R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right\} \sin 2z, \tag{4.6a}$$

where $$\frac{dH_2}{dt} + \left\{-\frac{2a^2R}{(1+\alpha^2)^2} g(t) + 2a_0 + 4\right\} H_2 = -\frac{1}{2}\frac{\alpha^2 R}{(1+\alpha^2)^2}\Theta_0^2(1-g(t))$$

and $$H_2(0) = 0.$$

Define the integrating factor $J_2(t)$,

$$J_2(t) = \exp\left\{(4+2a_0)\,t - \frac{2\alpha^2 R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right\}. \tag{4.6b}$$

We then have that

$$H_2(t) = -\frac{1}{2}\frac{\alpha^2 R}{(1+\alpha^2)^2}\Theta_0^2 J_2^{-1}(t)\int_0^t (1-g(s))\,J_2(s)\,ds. \tag{4.6c}$$

All corrections at third order are easily calculated with the exception of that one of $O(A^3 \cos \alpha x)$:

$$\left(\frac{\partial^2}{\partial z^2} - \alpha^2\right)^2 w_3 - \alpha^2 R(1-g(t))\,\theta_3 = 0, \tag{4.7a}$$

$$\left(-\frac{\partial}{\partial t} - 3a_0 + \frac{\partial^2}{\partial z^2} - \alpha^2\right)\theta_3 + w_3 = -a_2\theta_1 + w_1 T_{2z}, \tag{4.7b}$$

with $$\theta_3 = w_3 = w_{3zz} = 0 \quad \text{on} \quad z = 0, \pi \tag{4.7c}$$

and $$\theta_3 = w_3 = 0 \quad \text{at} \quad t = 0. \tag{4.7d}$$

We have simplified the right-hand side of (4.7$b$) by using the solutions (4.4).

When we treat the problem of the stability of time-independent states, $g(t) \equiv 0$. The above non-linear theory can be shown to emerge for formal perturbation theory (Eckhaus 1965) in the parameter $\delta = (R - R_L)^{\frac{1}{2}}$. The $O(\delta^3)$ terms are given in part by system (4.7) with the left-hand side modified as follows: $\partial\theta_3/\partial t$ is missing and $R$ is replaced by $R_L$. If one eliminates, say, $w_3$ from (4.7$a, b$), the left-hand side then represents a self-adjoint Fredholm operator subject to (4.7$c$).

The system (4.7) then only has solutions if the new right-hand side is orthogonal to $\theta_1$, the solution to the homogeneous problem. The constant $a_2$ is chosen to assure that this condition is satisfied. We thus suppress the replication terms, i.e. those terms on the right that have the $z$ dependence $\sin z$. However, in our problem the presence of the $\partial/\partial t$ prevents the operator on the left from being of Fredholm type.

In order to see the correct procedure for choosing $a_2(t)$, let us eliminate, say, $w_3$ between (4.7$a, b$) and obtain

$$\left(-\frac{\partial}{\partial t}-3a_0+\frac{\partial^2}{\partial z^2}-\alpha^2\right)\left(\frac{\partial^2}{\partial z^2}-\alpha^2\right)^2\theta_3+\alpha^2R(1-g(t))\,\theta_3 = v_1(t)\sin z+v_3(t)\sin 3z, \tag{4.8a}$$

where
$$v_1(t) = -(1+\alpha^2)^2\left\{a_2G_1+F_1H_2\exp\left[-\frac{2\alpha^2R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right]\right\} \tag{4.8b}$$

and
$$v_3(t) = F_1H_2(9+\alpha^2)^2\exp\left[-\frac{2\alpha^2R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right], \tag{4.8c}$$

with
$$\theta_3 = \theta_{3zz} = \theta_{3zzzz} = 0 \quad \text{on} \quad z = 0, \pi \tag{4.8d}$$

and
$$\theta_3 = 0 \quad \text{at} \quad t = 0. \tag{4.8e}$$

We seek a particular solution $\theta_3(t) = c_n(t)\sin nz$, $n = 1, 3$ for those terms on the right of (4.8$a$) containing $\sin nz$. We obtain

$$\frac{dc_n}{dt}+\left\{3a_0+(n^2+\alpha^2)-\frac{\alpha^2R}{(n^2+\alpha^2)^2}+\frac{\alpha^2R}{(n^2+\alpha^2)^2}g(t)\right\}c_n = -\frac{v_n(t)}{(n^2+\alpha^2)^2},$$

where the $v_n$ are bounded in time. We introduce the integrating factor $I_n(t)$,

$$I_n(t) = \exp\left\{\left[3a_0-\frac{\alpha^2}{(n^2+a^2)^2}\left(R-\frac{(n^2+\alpha^2)^3}{\alpha^2}\right)\right]t+\frac{\alpha^2R}{(n^2+\alpha^2)^2}\int_0^t g(s)\,ds\right\}$$

and find that

$$c_n(t) = -\frac{1}{(n^2+\alpha^2)^2}I_n^{-1}(t)\int_0^t v_n(s)I_n(s)\,ds+\text{const.}\,I_n^{-1}(t).$$

We shall show that $c_3(t)$ is bounded for all time showing that our statement above, that 'most' terms are easily treated, is valid.

When $n = 3$, and $R > R_L$,

$$I_3(t) = \exp\left\{\beta_3t+\gamma_3\int_0^t g(s)\,ds\right\},$$

where $\beta_3 > 0$ and $\beta_3$ remains $O(1)$ as $R\to R_L$. Hence for large $t$,

$$c_3(t) \sim \text{const.}\,I_3^{-1}(t)\int_0^t I_3(s)\,v_3(s)\,ds$$

so that $\lim_{t\to\infty} c_3(t) = \text{const.}\,\beta_3^{-1}$ and $c_3(t)$ is bounded for all time.

On the other hand, $c_1(t)$ becomes unbounded in time as $t\to\infty$. In this case if $R > R_L$,

$$I_1(t) = \exp\left\{\beta_1t+\gamma_1\int_0^t g(s)\,ds\right\},$$

where $\beta_1 > 0$ and $\beta_1 \to 0$ as $R \to R_L$. Hence

$$I_1^{-1}(t)\int_0^t v_1(s)I_1(s)\,ds \geqslant \inf|v_1(t)|\exp\left\{-\gamma_1\int_0^t g(\xi)\,d\xi\right\}\int_0^t \exp\left\{\gamma_1\int_0^s g(\xi)\,d\xi\right\}ds$$

$$\to \infty \quad \text{as} \quad t\to\infty \quad \text{and} \quad R \to R_L.$$

We must therefore choose $a_2(t)$ so that $v_1(t) \equiv 0$. This is precisely the requirement that would have to be met had $\partial/\partial t$ been absent and a Fredholm alternative argument been made. From (4.8*b*),

$$a_2(t) = -F_1(t)G_1^{-1}(t)H_2(t)\exp\left\{-\frac{2\alpha^2R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right\}. \tag{4.9}$$

## 5. Analysis of the amplitude equation

From §4, we have derived the following amplitude equation:

$$dY/dt = \gamma_0(t)Y - \gamma_2(t)Y^3 + \ldots, \tag{5.1a}$$

where
$$Y(t) = G_1(t)A(t), \tag{5.1b}$$

$$\gamma_0(t) = a_0 - \frac{\alpha^2R}{(1+\alpha^2)^2}g(t), \tag{5.1c}$$

$$\gamma_2(t) = \frac{1}{2}\left[\frac{\alpha^2R}{(1+\alpha^2)^2}\right]^2(1-g(t))J_2^{-1}(t)\int_0^t(1-g(s))J_2(s)\,ds, \tag{5.1d}$$

$$J_2(t) = \exp\left\{(4+2a_0)t - \frac{2\alpha^2R}{(1+\alpha^2)^2}\int_0^t g(s)\,ds\right\}. \tag{5.1e}$$

The initial condition is
$$Y(0) = \Theta_0. \tag{5.1f}$$

We first note that if $1-g(t) > 0$ for all time, then $\gamma_2(t) > 0$ for all time, independent of the sign of $R-R_L$. Thus, the non-linearities are always stabilizing.

If we define $K(t)$ as follows:

$$K(t) = \exp\left\{2\int_0^t \gamma_0(s)\,ds\right\}, \tag{5.2a}$$

the solution of (5.1) is given by

$$Y(t) = \left\{\Theta_0^{-2}K^{-1}(t) + 2K^{-1}(t)\int_0^t \gamma_2(s)K(s)\,ds\right\}^{-\frac{1}{2}}. \tag{5.2b}$$

### (*a*) *A monotone example*: $g(t) = \epsilon e^{-t}$

If one wishes to consider the stability of a layer of fluid whose upper boundary is impulsively cooled, the basic state (Foster 1965*a*, *b*) takes the form $\mathbf{v} \equiv 0$ with

$$\overline{T}(z,t) = -z - 2\sum_{n=1}^{\infty}(-1)^n\frac{\sin nz}{n}e^{-n^2t}.$$

The dominant term gives $\quad -\overline{T}_z(z,t) = 1 - 2e^{-t}\cos z.$

Thus, we might expect that if we let $g(t) = \epsilon e^{-t}$ in our model there may be some qualitative similarity with the above problem.

Equation (5.2) was thus examined with $g(t) = \epsilon e^{-t}$. It is easy to see that as $t \to \infty$, $Y^2(t) \to 8a_0[(1+\alpha^2)^2/\alpha^2 R]^2 \equiv Y_e^2$ when $R > R_L$ and $Y^2 \to 0$ as $t \to \infty$ when $R < R_L$. In the unstable case, $R > R_L$, $Y \to Y_e$ where $Y_e$ is the *identical amplitude* predicted when $g(t) \equiv 0$, the steady case (see Segel 1962 and note the different scaling). The approach to steady-state, however, is not the same. For example, in the steady case, $Y(t)$ grows to $Y_e$ monotonically. When $g(t) = \epsilon e^{-t}$, $Y(t)$ *decreases* from its initial value until $t = t_0$ where $\gamma_0(t_0) = 0$ approximately. (Here we have neglected the contribution due to the non-linearities since we presume that $Y(0)$ is small.) $t_0$ satisfies

$$\epsilon \frac{\alpha^2 R}{(1+\alpha^2)^2} e^{-t_0} = a_0$$

or

$$t_0 = \ln \epsilon + \ln \frac{R}{R - R_L}.$$

Hence since $R > R_L$, $t_0 \to \infty$ as $R \to R_L$. The initial decrease in disturbance amplitude can indeed occur in impulsive heating of fluid layer (Foster 1965*a*) and is due there to the finite diffusion time needed for the change in boundary temperature to effect an unstable temperature gradient.

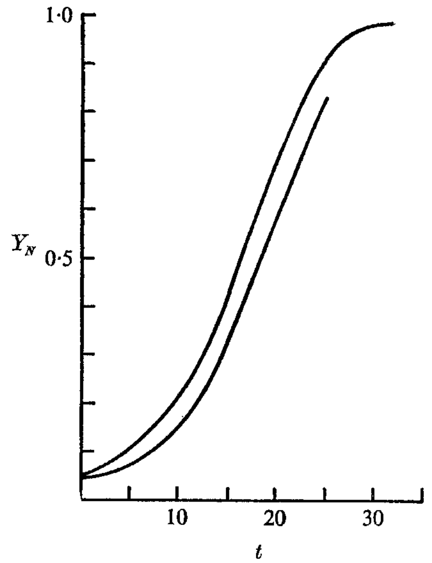


FIGURE 1. $Y_N$ *vs.* $t$ for $R = 1{\cdot}1R_L$, $\alpha^2 = 0{\cdot}5$; $g(t) = 0{\cdot}2\, e^{-t}$ (lower curve) and for $g(t) \equiv 0$ (upper curve).

A comparison of the cases $g(t) \equiv 0$ and $g(t) = \epsilon e^{-t}$ is given in figure 1. $Y_N(t) \equiv Y(t) Y_e^{-1}$ is plotted as a function of $t$ for $\alpha^2 = 0{\cdot}5$, $R_L = 6{\cdot}75$, $R = 1{\cdot}1R_L$ and $\epsilon = 0{\cdot}2$. For these values, $t_0 \doteq 0{\cdot}8$. Note that the initial decrease of $Y_N(t)$ in the time-dependent case slows the growth of $Y_N(t)$ for all time. Even though both curves ultimately approach unity, the time it takes the lower curve to reach a given value can be an order of magnitude larger than in the steady case for certain values of parameters.

In fact, it is easy to see that if $g(t) \to 0$ as $t \to \infty$, $Y(t) \to Y_e$ regardless of the form of $g(t)$. This validates, at least in our problem, the usual assumption made in the non-linear theory of steady states, viz. if one does a Bénard experiment by raising the temperature in small steps, the results can be compared to a theoretical analysis of the *steady state* problem. We see here that if one waits long enough, the *same finite amplitude* is attained.

### (b) *An oscillatory example*: $g(t) = \epsilon \sin(\omega t + \phi)$

We wish to see how time modulation influences the growth of $Y_N(t)$. We stress that due to the simplifications of our model, the critical value $R_L$ of $R$ according to linear theory is the same as without modulation. There is no 'Mathieu effect' to raise or lower the stability boundary (see Gresho & Sani 1970; Venezian 1969).

We take $g(t) = \epsilon \sin(\omega t + \phi)$. As before, the linearized amplitude equation has points $t = t_0$ at which $dY/dt = 0$. These are given by $\gamma_0(t_0) = 0$:

$$\epsilon \frac{\alpha^2 R}{(1+\alpha^2)^2} \sin(\omega t_0 + \phi) = a_0$$

or

$$\omega t_0 = -\phi + \sin^{-1}\left(\frac{R - R_L}{\epsilon R}\right).$$

Again the non-linearities are only stabilizing since if $1 - g(t) > 0$, then $\gamma_2(t) > 0$.

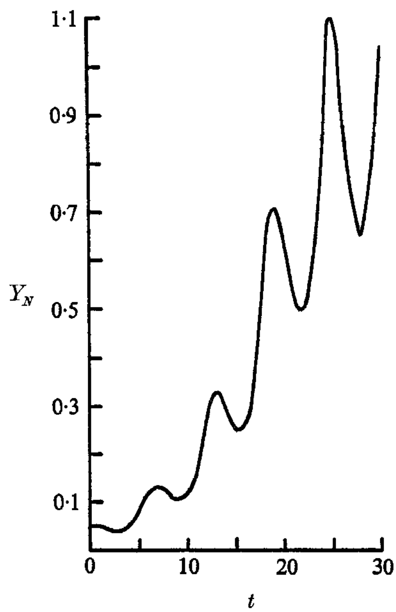


FIGURE 2. $Y_N$ *vs.* $t$ for $g(t) = 0{\cdot}2 \sin t$, $R = 1{\cdot}1 R_L$, $\alpha^2 = 0{\cdot}5$.

Figure 2 illustrates the case with $\epsilon = 0{\cdot}2$, $\omega = 1$, $\phi = 0$, $R = 1{\cdot}1 R_L$ and $\alpha^2 = 0{\cdot}5$. $Y_N(t) \equiv Y(t)/Y_e$ is plotted as a function of time. For the values given we find that

$$t_0 \doteq 0{\cdot}47 + 2n\pi \quad (n = 0, 1, 2, \ldots)$$

and

$$t_0 \doteq \pi - 0{\cdot}47 = 2n\pi \quad (n = 0, 1, 2, \ldots).$$

The modulated exponential growth is seen to approach a finite amplitude synchronous oscillation for long times. Had we considered a less restricted model, Mathieu effects might be present and the oscillation might well not have been synchronous.

## 6. Discussion and conclusions

Formal expansions of the type given in (3.3) can be applied to a whole class of problems. The expansion is essentially a power series in the small parameter $(R-R_L)^{\frac{1}{2}}$ which is a measure of $A_e \equiv \lim_{t\to\infty} A(t)$ if this exists or else in a properly defined bound on $A(t)$, e.g. $\sup_{t\in[0,\infty)} |A(t)|$. Whenever the system of governing partial differential equations is such that the $\theta_i$, $w_i$, $u_i$, $T_i$ are uniformly bounded in time and a finite secondary state exists, $R-R_L$ can be made small enough to make systems of the form (3.3) well defined. In the problem solved herein, it was sufficient that $\int_0^t g(s)\,ds$ be bounded for all finite time for a secondary state to exist.

The crucial observation motivating the present method is that in certain problems the time variation of the linearized solution could be separated into a product of two parts: a bounded function of $t$ and possibly the spatial variables times an exponential whose growth rate could be made arbitrarily close to zero by controlling the value of the external parameter $R$. The usual Stuart–Watson arguments are made by replacing the exponential by an amplitude function $A(t)$. $A(t)$ satisfies a non-linear ordinary differential equation whose coefficients $a_i(t)$ are found by requiring the solution to be uniformly valid in time. No assumption concerning the scale of the time variation of the basic state (say, quasi-steadiness) is necessary. The particular problem treated in this paper was simplified in two severe ways. We assumed that $P\to\infty$ and that the boundary conditions represented planar, stress-free boundaries. The first assumption made the governing equations first order in time while the second allowed $\theta_1$, $w_1$, $u_1$ to be found as separable functions of $z$ and $t$. Let us see how the relaxation of these assumptions affects the procedure.

It is worth noting here that for large $P$ (3.3$e$) closely approximates (3.4) in a singular perturbation sense. If we define $\lambda = (1+P)^{-1}(1+\alpha^2)^{-1}$, the parameter that multiplies $d^2A/dt^2$ in (3.4), it is easy to see that the retention of $\lambda(d^2A/dt^2)$ for small $\lambda$ gives rise to a 'boundary layer' of thickness $O(\lambda)$ near $t=0$. This layer allows a condition on $dA/dt(0)$ to be satisfied. For large times (3.3$e$) gives the correct qualitative behaviour.

(i) Let $P$ be finite but let the free–free boundary conditions still apply. The linearized problem after the $x$ variation has been separated out has solutions (see (3.1) and (3.2))

$$(\hat{\theta}_1, \hat{w}_1) = (G_1(t), F_1(t)) \exp(a_0 t) \sin z,$$

where
$$P^{-1}(dF_1/dt)(1+\alpha^2) + (a_0 P^{-1} + (1+\alpha^2)^2) F_1 - \alpha^2 R(1-g(t)) G_1 = 0 \quad (6.1a)$$

and
$$-dG_1/dt - (a_0 + 1 + \alpha^2) G_1 + F_1 = 0. \quad (6.1b)$$

This system of linear equations has variable coefficients. For example, when $g(t) = \epsilon \sin(\omega t + \phi)$, (6.1) can be transformed to a Matheiu equation and the solutions are periodic functions of $t$ (if the Floquet exponents are simple) dependent on $(a_0, \epsilon, \omega, \phi, R, P)$ and $a_0 = a_0(\epsilon, \omega, \phi, R, P)$ is the Floquet exponent. The above outlined *procedure* goes through but the amplitude equation (3.3*e*) now takes the form (3.4) (see §3). The spirit used in the expansion is the same; the answer is different. It is to be expected that (3.4) would have some features of its solution different from that of (3.3*e*) since it itself can be transformed into a non-linear Mathieu equation. For $g(t)$ other than above, we would require a solution $(G_1, F_1)$ bounded in time and then again (3.4) would be the representative amplitude equation.

(ii) Let $P \to \infty$ but let the boundaries be rigid planes. The linearized problem has the form (3.1*a, b*) but with the boundary conditions

$$\hat{w}_1 = \hat{w}_{1z} = \hat{\theta}_1 = 0 \quad \text{on} \quad z = 0, \pi. \tag{6.2}$$

Due to (6.2), the $z$ dependence is no longer separable from the $t$ dependence. We must seek solutions of the form

$$(\hat{w}_1, \hat{\theta}_1) = (F_1(z,t), G_1(z,t)) \exp(a_0 t),$$

where $F_1$ and $G_1$ are bounded in $t$. These can be found (if they exist) using, say) Galerkin's method. This method is outlined in the appendix for the more general case of $P$ finite. Having solved the linear problem, the outlined procedure for the non-linear problem seems still to be valid. The $a_i(t)$ are again chosen to make the solution uniformly valid in time. The amplitude equation that results again has the form (3.3*e*).

The object of this work is to communicate the spirit of the approach which is much the same as has been used successfully in treating the stability of steady states (Segel 1966). The decomposition of the time dependence of the linear theory eigenfunctions mentioned above makes this possible.

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## Appendix: Formal solution of the linearized problem

Let us consider a generalized version of the linearized Bénard problem. The disturbance equations can be written (Davis 1969*a*) as follows:

$$\left(-P^{-1}\frac{\partial}{\partial t} + L\right) Lw - \alpha^2 R^{\frac{1}{2}} S_1(z,t)\,\phi = 0, \tag{A 1a}$$

$$\left(-\frac{\partial}{\partial t} + L\right)\phi + R^{\frac{1}{2}} S_2(z,t)\,w = 0, \tag{A 1b}$$

where $L \equiv D^2 - \alpha^2$ and $R^{\frac{1}{2}}\theta = \phi$. When $S_1(z,t) \equiv 1 - g(t)$ and $S_2(z,t) \equiv 1$, equations (A 1) reduce to the example treated in the text concerning a modulated gravity field. When $S_1$ and $S_2$ have other forms, heat sources or time-dependent boundary temperatures can be modelled. When $P = 1$, the small gap Couette flow instability leading to Taylor vortices is included.

The boundary conditions are the following:

$$w = \theta = 0 \quad \text{on} \quad z = 0, \pi \tag{A 2a}$$

and
$$\left.\begin{cases} D^2w(0) = D^2w(\pi) = 0, & \text{for free–free boundaries,} \\ Dw(0) = Dw(\pi) = 0, & \text{for rigid–rigid boundaries,} \\ Dw(0) = D^2w(\pi) = 0, & \text{for rigid–free boundaries.} \end{cases}\right\} \tag{A 2b}$$

We will attempt to solve (A 1) subject to the boundary conditions (A 2$a$) and one of (A 2$b$) using the Galerkin method. (For a similar approach, see Grosch & Salwen 1968.)

Consider the functions $\{\Phi_n\}$ such that

$$L^2\Phi_n + \lambda_n L\Phi_n = 0, \tag{A 3}$$

subject to the same boundary condition as $w$. It is easy to show that $\lambda_n > 0$ for all $n$ and that

$$-\langle L\Phi_n, \Phi_k\rangle = \delta_{nk}, \tag{A 4}$$

where
$$\langle \alpha, \beta\rangle \equiv \int_0^\pi \alpha\beta\, dz.$$

Consider the functions $\{\Psi_n | \Psi_n = (2/\pi)^{\frac{1}{2}} \sin nz\}$ which satisfy the same boundary conditions as $\theta$.

We solve the partial differential equations (A 1) by writing

$$w(z,t) = \sum_{n=1}^{\infty} a_n(t)\, \Phi_n(z), \tag{A 5a}$$

$$\theta(z,t) = \sum_{n=1}^{\infty} b_n(t)\, \Psi_n(z). \tag{A 5b}$$

We substitute the first $N$ terms of (A 5) into (A 1$a$), multiply by $\Phi_k$ and integrate from $z = 0$ to $z = \pi$ and obtain:

$$P^{-1}\dot{a}_k + \lambda_k a_k - \alpha^2 R^{\frac{1}{2}} \sum_{n=1}^{N} b_n \langle S_1 \Psi_n, \Phi_k\rangle = 0 \quad (k = 1, 2, \ldots, N). \tag{A 6a}$$

We substitute the first $N$ terms of (A 5) into (A 1$b$), multiply by $\Psi_k$ and integrate from $z = 0$ to $z = \pi$ and obtain

$$\dot{b}_k + (k^2 + \alpha^2) b_k - R^{\frac{1}{2}} \sum_{n=1}^{N} a_n \langle S_2 \Phi_n, \Psi_k\rangle = 0 \quad (k = 1, 2, \ldots, N). \tag{A 6b}$$

Equations (A 6) have been simplified using the defining equations for $\Phi_n$ and $\Psi_n$ and their orthogonality properties.

Let
$$M(t) = (\langle S_1(z,t)\, \Psi_n, \Phi_k\rangle),$$
$$N(t) = (\langle S_2(z,t)\, \Phi_n, \Psi_k\rangle),$$
$$D_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_N \end{pmatrix},$$

$$D_2 = \begin{pmatrix} 1+\alpha^2 & & & 0 \\ & 4+\alpha_2 & & \\ & & \ddots & \\ 0 & & & N^2+\alpha^2 \end{pmatrix},$$

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix}.$$

We then have from (A 6) that

$$\frac{d}{dt}\begin{pmatrix} P^{-1}A \\ B \end{pmatrix} = -\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} + R^{\frac{1}{2}}\begin{pmatrix} 0 & \alpha^2 M(t) \\ N(t) & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}. \tag{A 7}$$

One possibility is that exhibited in the text where $M(t) = -g(t)\,(\langle\Psi_n, \Phi_k\rangle)$ and $N(t)$ is independent of time. When $g(t)$ is periodic and the Floquet exponents are simple, then there exist solutions of (A 7) of the form

$$(a_1, b_1) = e^{\mu_1 t}(F_i^{(1)}(t), F_i^{(2)}(t)) \quad (i = 1, 2, \ldots, N)$$

where $\mu_i$ is constant and $F_i^{(1)}$, $F_i^{(2)}$ are periodic in time at least when $N$ is finite (Coddington & Levinson 1955). The approximate versions (A 5) can then be written as follows

$$(w(z,t), \theta(z,t)) = e^{\sigma t}(G_1(z,t), G_2(z,t)), \tag{A 8}$$

where $\sigma = \max\limits_j\,(\mu_j)$. Thus $G_1$ and $G_2$ are bounded in time and the value of $\sigma$ is controllable by varying the value of $R$. Note that this is valid for arbitrary $P$, not only $P \to \infty$.

The non-linear stability theory proposed is valid as long as a solution to system (A 1) exists of the form (A 8). Clearly $M(t)$ and $N(t)$ need not be periodic but could have more general form. For example, when the layer sustains time-dependent heating of a horizontal boundary (Foster 1965*a*), $N(t)$ is independent of time while $M(t)$ involves sums of exponentially decaying sinusoids.

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